# DDCO

## UNIT-1

CLASS NOTES

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# introduction

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- knowledge of hardware essential for efficient software

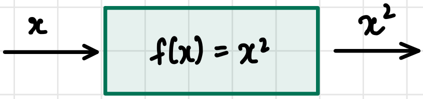
## Moore's Law

- ~30 years ago, supercomputers needed liquid nitrogen
- Every ~18 months, no. of transistors per chip area doubles
- Transistor speed 2x, power consumption ½x.
- Moore's Law is now slowing down
- Therefore, to improve performance, deeper knowledge of hardware needed for good software
- Targeted hardware accelerators developed
- Google has TensorFlow & Tensor Processing Unit accelerators

# BOOLEAN FUNCTIONS

## Mathematical Functions

- Example: parabola
- Domain & range set of real numbers
- Can be specified on Cartesian plane
- Specify function as a box

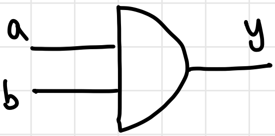



## Boolean Functions

- Example: AND function/gate
- Domain & range: {0,1}
- Specify function using truth table

| a | b | y |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

- Specify as logic gate (black box)

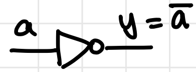



TODO: redraw

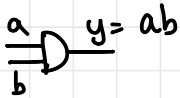
# BASIC FUNCTIONS / LOGIC GATES

1) BUFFER

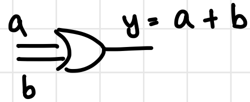

2) NOT



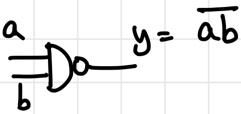

3) AND



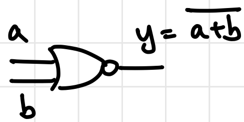

4) OR



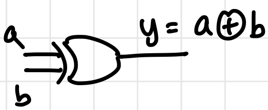

5) NAND



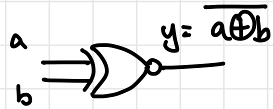

6) NOR




7) XOR




8) XNOR




- Number of rows for a n-input TT = $2^n$
- How many different 1-input boolean functions can exist? ($2^{2^1}$)

## Layers of Abstraction

- Logic minimisation: truth table
- Logic design level: component in logic circuit
- CMOS VLSI design: transistor level circuit diagram with digital switches (on or off)
- VLSI layout level (silicon, metal)
- VLSI fabrication level – chip (microprocessor)

# Boolean Algebra & Identities

## Logic circuits

- Multiple logic gates
- O/P of one gate connected to I/P of another
- Represent Boolean functions
- Look at Perfect Induction

## Boolean Formulas

- Boolean constants (0 and 1) are Boolean formulas
- Boolean variables (like x) are Boolean formulas
- If P & Q and Boolean formulas,

Precedence
1. $\overline{P}$
2. $P \cdot Q$
3. $P + Q$

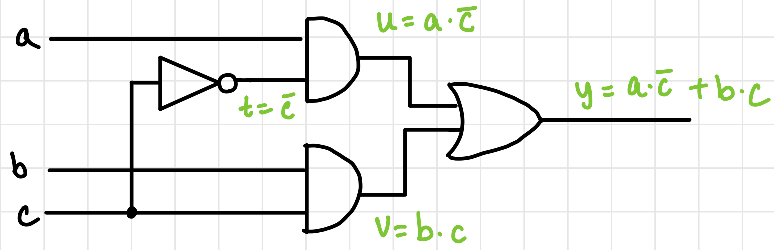
Boolean formulas

Q: How is $((a \cdot \overline{c}) + (b \cdot c))$ a Boolean formula?
1. a, b & c are Boolean formulas
2. $\overline{c}$ is a Boolean formula
3. $a \cdot \overline{c}$ and $b \cdot c$ are Boolean formulas
4. $a \cdot \overline{c} + b \cdot c$ is a Boolean formula

## Boolean Formulas & Logic Circuits

1. Constants & variables are inputs to logic gates
2. $\cdot$ operator means AND
3. $+$ operator means OR
4. $\overline{\phantom{x}}$ operator means NOT

Q: Convert $((a\cdot\bar{c}) + (b\cdot c))$ to a logic circuit




## COMBINATIONAL LOGIC CIRCUITS

- Circuits that can be represented by Boolean formulas
- Unlike sequential logic circuits (with feedback)

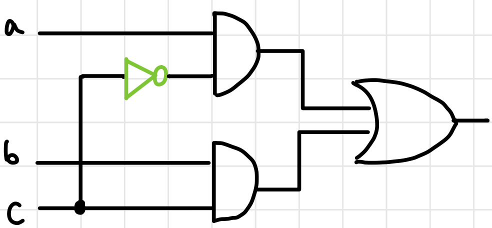
* HDL (hardware descriptive language) not in this course.

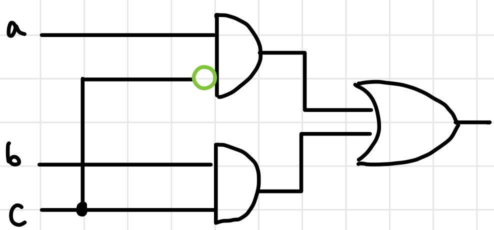
## BOOLEAN ALGEBRA & IDENTITIES

- Boolean functions can be represented as truth tables, combinational logic circuits and Boolean formulas
- Multiple Boolean functions & CLCs for Boolean formula

### Note - Bubbles

- NOT gates can be replaced with a bubble



# Boolean Algebra

1. Set: $\{0, 1\}$
2. Operations: AND, OR, NOT
3. Identity element: (0 – OR, 1 – AND)
4. Laws: commutative, associative, distributive

## Boolean Identities

### 1. Commutative

$$a \cdot b = b \cdot a$$
$$a + b = b + a$$

dual

### 2. Associative

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$
$$(a + b) + c = a + (b + c)$$

dual

### 3. Distributive

$$a \cdot (b + c) = a \cdot b + a \cdot c$$
$$a + (b \cdot c) = (a + b) \cdot (a + c)$$

dual

### 4. De Morgan's

$$\overline{(a + b)} = \bar{a} \cdot \bar{b}$$
$$\overline{(a \cdot b)} = \bar{a} + \bar{b}$$

dual

### 5. Idempotency

$$a \cdot a = a$$
$$a + a = a$$

6. Identity

$$a \cdot 1 = a$$
$$a + 0 = a$$

7. Boundedness

$$a \cdot 0 = 0$$
$$a + 1 = 1$$

8. Complement

$$a \cdot \bar{a} = 0$$
$$a + \bar{a} = 1$$

9. Absorption

$$a + a \cdot b = a$$
$$a \cdot (a + b) = a$$

10. Involution

$$\bar{\bar{a}} = a$$

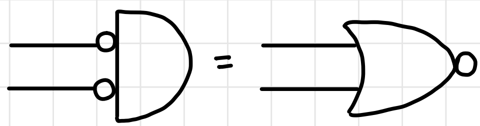
11. Useful Identity

$$a + \bar{a} \cdot b = a + b = (a + \bar{a}) \cdot (a + b) = a + b$$
$$a \cdot (\bar{a} + b) = a \cdot b = a \cdot \bar{a} + a \cdot b = a \cdot b$$

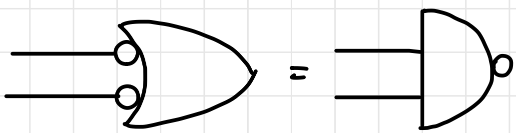
# MULTIPLE INPUTS

- Due to associativity, $a+b+c$ & $a \cdot b \cdot c$ require no brackets and the order does not matter.
- Using DeMorgan's Laws

$$\bar{a} \cdot \bar{b} = \overline{a+b}$$



$$\bar{a} + \bar{b} = \overline{a \cdot b}$$



## logic minimisation & K-MAPS

- construct Boolean formula / CLC using truth table
- using smallest two-level sum of Products formula

### TERMINOLOGY

- Literal — Bool variable / complement (eg: $\bar{c}$)
- Product — AND of two or more literals (eg: $ab\bar{c}$)
- Minterm — product involving all inputs to Boolean function (such that minterm product = 1)

| $a$ | $b$ | $c$ | minterm | name |
|---|---|---|---|---|
| 0 | 0 | 0 | $\bar{a}\bar{b}\bar{c}$ | $m_0$ |
| 0 | 0 | 1 | $\bar{a}\bar{b}c$ | $m_1$ |
| 0 | 1 | 0 | $\bar{a}b\bar{c}$ | $m_2$ |
| 0 | 1 | 1 | $\bar{a}bc$ | $m_3$ |
| 1 | 0 | 0 | $a\bar{b}\bar{c}$ | $m_4$ |
| 1 | 0 | 1 | $a\bar{b}c$ | $m_5$ |
| 1 | 1 | 0 | $ab\bar{c}$ | $m_6$ |
| 1 | 1 | 1 | $abc$ | $m_7$ |

lowercase m

names of the minterms

- sum — OR of two or more literals (eg: $a+\bar{b}+c$)
- Maxterm — sum involving all inputs to Boolean function (such that maxterm sum = 0)

| $a$ | $b$ | $c$ | maxterm | name |
|---|---|---|---|---|
| 0 | 0 | 0 | $a+b+c$ | $M_0$ |
| 0 | 0 | 1 | $a+b+\bar{c}$ | $M_1$ |
| 0 | 1 | 0 | $a+\bar{b}+c$ | $M_2$ |
| 0 | 1 | 1 | $a+\bar{b}+\bar{c}$ | $M_3$ |
| 1 | 0 | 0 | $\bar{a}+b+c$ | $M_4$ |
| 1 | 0 | 1 | $\bar{a}+b+\bar{c}$ | $M_5$ |
| 1 | 1 | 0 | $\bar{a}+\bar{b}+c$ | $M_6$ |
| 1 | 1 | 1 | $\bar{a}+\bar{b}+\bar{c}$ | $M_7$ |

uppercase M

names of maxterms

## Sum of Products

- sum of all minterms corresponding to a 1 output

- eg:

| a | b | c | y | minterm | name |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | $\bar{a}\bar{b}\bar{c}$ | $m_0$ |
| 0 | 0 | 1 | 0 | $\bar{a}\bar{b}c$ | $m_1$ |
| 0 | 1 | 0 | 0 | $\bar{a}b\bar{c}$ | $m_2$ |
| 0 | 1 | 1 | 1 | $\bar{a}bc$ | $m_3$ |
| 1 | 0 | 0 | 1 | $a\bar{b}\bar{c}$ | $m_4$ |
| 1 | 0 | 1 | 0 | $a\bar{b}c$ | $m_5$ |
| 1 | 1 | 0 | 1 | $ab\bar{c}$ | $m_6$ |
| 1 | 1 | 1 | 1 | $abc$ | $m_7$ |

- SOP form Boolean formula:

$$y = \bar{a}bc + a\bar{b}\bar{c} + ab\bar{c} + abc$$

$$y = m_3 + m_4 + m_6 + m_7$$

- sigma ($\Sigma$) notation

$$y = \Sigma(m_3, m_4, m_6, m_7)$$

$$y = \Sigma(3, 4, 6, 7)$$

- Boolean function f

$$y = f(a, b, c) = \Sigma(m_3, m_4, m_6, m_7)$$

- Can use 4 3-input AND gates and 1 4-input OR gate to construct CLC

# Product of Sums

- product of all maxterms corresponding to a 0 output

- eg:

| $a$ | $b$ | $c$ | $y$ | maxterm | name |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | $a+b+c$ | $M_0$ |
| 0 | 0 | 1 | 0 | $a+b+\overline{c}$ | $M_1$ |
| 0 | 1 | 0 | 0 | $a+\overline{b}+c$ | $M_2$ |
| 0 | 1 | 1 | 1 | $a+\overline{b}+\overline{c}$ | $M_3$ |
| 1 | 0 | 0 | 1 | $\overline{a}+b+c$ | $M_4$ |
| 1 | 0 | 1 | 0 | $\overline{a}+b+\overline{c}$ | $M_5$ |
| 1 | 1 | 0 | 1 | $\overline{a}+\overline{b}+c$ | $M_6$ |
| 1 | 1 | 1 | 1 | $\overline{a}+\overline{b}+\overline{c}$ | $M_7$ |

- POS from Boolean formula

$$y = (a+b+c)\cdot(a+b+\overline{c})\cdot(a+\overline{b}+c)\cdot(\overline{a}+b+\overline{c})$$

$$y = M_0 M_1 M_2 M_5$$

- Pi ($\Pi$) notation

$$y = \Pi(M_0, M_1, M_2, M_5)$$

$$y = \Pi(0,1,2,5)$$

- Boolean function f

$$y = f(a,b,c) = \Pi(0,1,2,5)$$

# Logic Minimisation using Identities

- Consider $y = \bar{a}bc + a\bar{b}\bar{c} + ab\bar{c} + abc$
- Minimise using identities

$$y = \bar{a}bc + a\bar{b}\bar{c} + ab\bar{c} + abc$$
$$= bc(a+\bar{a}) + a\bar{c}(b+\bar{b})$$
$$= bc + a\bar{c}$$

**Q: Prove that $ab + bc + a\bar{c} = bc + a\bar{c}$**

$$ab + bc + a\bar{c}$$
$$= ab(c+\bar{c}) + bc + a\bar{c}$$
$$= abc + ab\bar{c} + bc + a\bar{c}$$
$$= bc(a+1) + a\bar{c}(b+1)$$
$$= bc + a\bar{c}$$

$$= ab + \bar{a}bc + a\bar{b}\bar{c}$$
$$= a(b + \bar{b}\bar{c}) + \bar{a}bc$$
$$= a(b+\bar{c}) + \bar{a}bc$$
$$= ab + a\bar{c} + \bar{a}bc$$
$$= b(a + \bar{a}c) + a\bar{c}$$
$$= b(a+c) + a\bar{c}$$
$$= bc + a\bar{c} + ab$$

**Q: Minimise $(a+b+c)(a+b+\bar{c})(a+\bar{b}+c)(\bar{a}+b+\bar{c})$**

$$\left[(a+b) + (a+b)\bar{c} + (a+b)c\right](a+\bar{b}+c)(\bar{a}+b+\bar{c})$$
$$= (a+b)(a\cancel{\bar{a}} + ab + a\bar{c} + \bar{b}\bar{a} + \cancel{\bar{b}}b + \bar{b}\bar{c} + c\bar{a} + cb + 0)$$
$$= (a+b)(ab + a\bar{c} + \bar{a}\bar{b} + \bar{b}\bar{c} + c\bar{a} + bc)$$
$$= ab + a\bar{c} + a\bar{b}\bar{c} + abc + ab + ab\bar{c} + \bar{a}bc + bc$$
$$= ab(1+c) + a\bar{c}(1+\bar{b}) + ab\bar{c} + bc(1+\bar{a})$$
$$= ab + a\bar{c} + bc + ab\bar{c}$$
$$= ab + a\bar{c} + bc$$
$$= abc + ab\bar{c} + a\bar{c} + bc$$
$$= bc + a\bar{c}$$

# K-Maps

- Named after Maurice Karnaugh
- Minterms that differ in one literal must be adjacent

## IMPLICANTS

- K-Map area composed of squares containing 1s
- Area is square/rectangular (wraparound allowed)
- No. of squares in area is power of 2
- Each implicant → product of literals

### Prime Implicants

- Implicant with largest no. of squares obeying the rules

### Essential prime Implicant

- Prime implicant containing square not in any other prime implicant

- K map method: includes all prime implicants such that each 1 square is covered and formula is minimal.

- For this TT

| a | b | c | y | minterm | |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | $\bar{a}\bar{b}\bar{c}$ | 0 |
| 0 | 0 | 1 | 0 | $\bar{a}\bar{b}c$ | 1 |
| 0 | 1 | 0 | 0 | $\bar{a}b\bar{c}$ | 2 |
| 0 | 1 | 1 | 1 | $\bar{a}bc$ | 3 |
| 1 | 0 | 0 | 1 | $a\bar{b}\bar{c}$ | 4 |
| 1 | 0 | 1 | 0 | $a\bar{b}c$ | 5 |
| 1 | 1 | 0 | 1 | $ab\bar{c}$ | 6 |
| 1 | 1 | 1 | 1 | $abc$ | 7 |

| a \ bc | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 0 | 0 $\bar{a}\bar{b}\bar{c}$<br>0 | 1 $\bar{a}\bar{b}c$<br>0 | 3 $\bar{a}bc$<br>1 | 2 $\bar{a}b\bar{c}$<br>0 |
| 1 | 4 $a\bar{b}\bar{c}$<br>1 | 5 $a\bar{b}c$<br>0 | 7 $abc$<br>1 | 6 $ab\bar{c}$<br>1 |

## Example 1

1. Prime implicants — green
2. Essential — blue
3. Required — red

| a \ bc | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 0 | $0_{0}$ | $0_{1}$ | $1_{3}$ | $0_{2}$ |
| 1 | $1_{4}$ | $0_{5}$ | $1_{7}$ | $1_{6}$ |

| a \ bc | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 0 | $0_{0}$ | $0_{1}$ | $1_{3}$ | $0_{2}$ |
| 1 | $1_{4}$ | $0_{5}$ | $1_{7}$ | $1_{6}$ |

| | a \ bc | $\bar{b}\bar{c}$ 00 | $\bar{b}c$ 01 | $bc$ 11 | $b\bar{c}$ 10 |
|---|---|---|---|---|---|
| $\bar{a}$ | 0 | $0_{0}$ | $0_{1}$ | $1_{3}$ | $0_{2}$ |
| $a$ | 1 | $1_{4}$ | $0_{5}$ | $1_{7}$ | $1_{6}$ |

(3 & 4 only covered by these 2)

SOP: $a\bar{c} + bc$ (unchanging terms)

## Example 2

| a \ bc | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 0 | $0_{0}$ | $1_{1}$ | $1_{3}$ | $0_{2}$ |
| 1 | $1_{4}$ | $1_{5}$ | $1_{7}$ | $1_{6}$ |

| a \ bc | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 0 | $0_{0}$ | $1_{1}$ | $1_{3}$ | $0_{2}$ |
| 1 | $1_{4}$ | $1_{5}$ | $1_{7}$ | $1_{6}$ |

| a \ bc | $\bar{b}\bar{c}$ 00 | $\bar{b}c$ 01 | $bc$ 11 | $b\bar{c}$ 10 |
|---|---|---|---|---|
| $\bar{a}$ 0 | $0_{0}$ | $1_{1}$ | $1_{3}$ | $0_{2}$ |
| $a$ 1 | $1_{4}$ | $1_{5}$ | $1_{7}$ | $1_{6}$ |

(1,3 & 4,6)

SOP: $c + a = f(a,b,c)$

## Example 3

| a \ bc | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 0 | $0_{0}$ | $1_{1}$ | $0_{3}$ | $1_{2}$ |
| 1 | $0_{4}$ | $1_{5}$ | $1_{7}$ | $1_{6}$ |

| a \ bc | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 0 | $0_{0}$ | $1_{1}$ | $0_{3}$ | $1_{2}$ |
| 1 | $0_{4}$ | $1_{5}$ | $1_{7}$ | $1_{6}$ |

| a \ bc | $\bar{b}\bar{c}$ 00 | $\bar{b}c$ 01 | $bc$ 11 | $b\bar{c}$ 10 |
|---|---|---|---|---|
| $\bar{a}$ 0 | $0_{0}$ | $1_{1}$ | $0_{3}$ | $1_{2}$ |
| $a$ 1 | $0_{4}$ | $1_{5}$ | $1_{7}$ | $1_{6}$ |

| a \ bc | $\bar{b}\bar{c}$ 00 | $\bar{b}c$ 01 | $bc$ 11 | $b\bar{c}$ 10 |
|---|---|---|---|---|
| $\bar{a}$ 0 | $0_{0}$ | $1_{1}$ | $0_{3}$ | $1_{2}$ |
| $a$ 1 | $0_{4}$ | $1_{5}$ | $1_{7}$ | $1_{6}$ |

SOP = $\bar{b}c + b\bar{c} + ca$

SOP = $\bar{b}c + b\bar{c} + ba$

(no unique solution)

# Example 4

| ab \ cd | $\bar c\bar d$ 00 | $\bar c d$ 01 | $cd$ 11 | $c\bar d$ 10 |
|---|---|---|---|---|
| $\bar a\bar b$ 00 | 1 (0) | (1) | (3) | 1 (2) |
| $\bar a b$ 01 | (4) | 1 (5) | 1 (7) | (6) |
| $ab$ 11 | (12) | 1 (13) | 1 (15) | (14) |
| $a\bar b$ 10 | 1 (8) | (9) | (11) | 1 (10) |


| ab \ cd | $\bar c\bar d$ 00 | $\bar c d$ 01 | $cd$ 11 | $c\bar d$ 10 |
|---|---|---|---|---|
| $\bar a\bar b$ 00 | 1 (0) | (1) | (3) | 1 (2) |
| $\bar a b$ 01 | (4) | 1 (5) | 1 (7) | (6) |
| $ab$ 11 | (12) | 1 (13) | 1 (15) | (14) |
| $a\bar b$ 10 | 1 (8) | (9) | (11) | 1 (10) |



| ab \ cd | $\bar c\bar d$ 00 | $\bar c d$ 01 | $cd$ 11 | $c\bar d$ 10 |
|---|---|---|---|---|
| $\bar a\bar b$ 00 | 1 (0) | (1) | (3) | 1 (2) |
| $\bar a b$ 01 | (4) | 1 (5) | 1 (7) | (6) |
| $ab$ 11 | (12) | 1 (13) | 1 (15) | (14) |
| $a\bar b$ 10 | 1 (8) | (9) | (11) | 1 (10) |


SOP = $db + \bar d\bar b$

# Example 5

| a | b | y |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 0 |
| 1 | 0 | 1 |
| 1 | 1 | 1 |

| a \ b | $\bar b$ 0 | $b$ 1 |
|---|---|---|
| $\bar a$ 0 | 1 (0) | 0 (1) |
| $a$ 1 | 1 (2) | 1 (3) |

SOP = $\bar b + a$

# DON'T CARES

- If we know for a fact that certain input combinations cannot occur in a logic gate, we place a don't care
- The logic circuit will never receive that input combination, so we don't care if it's a 0 or a 1

## Example 6

green - required (? = X)

| a \ bc | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 0 | $1_{0}$ | $1_{1}$ | $0_{3}$ | $0_{2}$ |
| 1 | $1_{4}$ | $1_{5}$ | $X_{7}$ | $1_{6}$ |

Do for both 0 and 1

| | a \ bc | $\bar{b}\bar{c}$ 00 | $\bar{b}c$ 01 | $bc$ 11 | $b\bar{c}$ 10 |
|---|---|---|---|---|---|
| $\bar{a}$ | 0 | $1_{0}$ | $1_{1}$ | $0_{3}$ | $0_{2}$ |
| $a$ | 1 | $1_{4}$ | $1_{5}$ | $0_{7}$ | $1_{6}$ |

| | a \ bc | $\bar{b}\bar{c}$ 00 | $\bar{b}c$ 01 | $bc$ 11 | $b\bar{c}$ 10 |
|---|---|---|---|---|---|
| $\bar{a}$ | 0 | $1_{0}$ | $1_{1}$ | $0_{3}$ | $0_{2}$ |
| $a$ | 1 | $1_{4}$ | $1_{5}$ | $1_{7}$ | $1_{6}$ |

$$SOP = y = \bar{b} + \bar{c}a$$

$$\boxed{SOP = y = \bar{b} + a}$$

- Whichever results in a smaller Boolean formula is used

## Example 7

$$f(a,b,c,d) = \Sigma(m_3, m_4, m_5, m_7, m_9, m_{13}, m_{14}, m_{15})$$

| ab \ cd | $\bar{c}\bar{d}$ 00 | $\bar{c}d$ 01 | $cd$ 11 | $c\bar{d}$ 10 |
|---|---|---|---|---|
| $\bar{a}\bar{b}$ 00 | $0_{0}$ | $0_{1}$ | $1_{3}$ | $0_{2}$ |
| $\bar{a}b$ 01 | $1_{4}$ | $1_{5}$ | $1_{7}$ | $0_{6}$ |
| $ab$ 11 | $0_{12}$ | $1_{13}$ | $1_{15}$ | $1_{14}$ |
| $a\bar{b}$ 10 | $0_{8}$ | $1_{9}$ | $0_{11}$ | $0_{10}$ |

$$y = \bar{c}\bar{a}b + cd\bar{a} + abc + \bar{c}da$$

## Example 8

$$f(a,b,c,d) = \Sigma(0,1,2,6,8,9,10)$$

| ab \ cd | $\bar{c}\bar{d}$ 00 | $\bar{c}d$ 01 | $cd$ 11 | $c\bar{d}$ 10 |
|---|---|---|---|---|
| $\bar{a}\bar{b}$ 00 | $1_{0}$ | $1_{1}$ | $0_{3}$ | $1_{2}$ |
| $\bar{a}b$ 01 | $0_{4}$ | $0_{5}$ | $0_{7}$ | $1_{6}$ |
| $ab$ 11 | $0_{12}$ | $0_{13}$ | $0_{15}$ | $0_{14}$ |
| $a\bar{b}$ 10 | $1_{8}$ | $1_{9}$ | $0_{11}$ | $1_{10}$ |

$$y = \bar{c}\bar{b} + \bar{a}c\bar{d} + \bar{b}c\bar{d}$$

## Example 9

$$f(a,b,c,d) = \Sigma(0,2,3,5,7,8,9,10,11,13,15)$$

| ab \ cd | $\bar{c}\bar{d}$ 00 | $\bar{c}d$ 01 | $cd$ 11 | $c\bar{d}$ 10 |
|---|---|---|---|---|
| $\bar{a}\bar{b}$ 00 | 1 (0) | (1) | 1 (3) | 1 (2) |
| $\bar{a}b$ 01 | (4) | 1 (5) | 1 (7) | (6) |
| $ab$ 11 | (12) | 1 (13) | 1 (15) | (14) |
| $a\bar{b}$ 10 | 1 (8) | 1 (9) | 1 (11) | 1 (10) |

$$y = bd + \bar{b}c + a\bar{b} + \bar{d}\bar{b}$$

not the only solution

## Example 10

don't care

$$f(a,b,c,d) = \Sigma(1,3,10) + \Sigma_d(0,2,8,12)$$

| ab \ cd | $\bar{c}\bar{d}$ 00 | $\bar{c}d$ 01 | $cd$ 11 | $c\bar{d}$ 10 |
|---|---|---|---|---|
| $\bar{a}\bar{b}$ 00 | X (0) | 1 (1) | 1 (3) | X (2) |
| $\bar{a}b$ 01 | 0 (4) | 0 (5) | 0 (7) | 0 (6) |
| $ab$ 11 | X (12) | 0 (13) | 0 (15) | 0 (14) |
| $a\bar{b}$ 10 | X (8) | 0 (9) | 0 (11) | 1 (10) |

$$y = \bar{d}\bar{b} + \bar{a}\bar{b}$$

# Example 11

$$f(w,x,y,z) = \Sigma(1,3,7,11,15) + \Sigma_d(0,2,5)$$

| wx \ yz | $\bar{y}\bar{z}$ 00 | $\bar{y}z$ 01 | $yz$ 11 | $y\bar{z}$ 10 |
|---|---|---|---|---|
| $\bar{w}\bar{x}$ 00 | X (0) | 1 (1) | 1 (3) | X (2) |
| $\bar{w}x$ 01 | (4) | X (5) | 1 (7) | (6) |
| $wx$ 11 | (12) | (13) | 1 (15) | (14) |
| $w\bar{x}$ 10 | (8) | (9) | 1 (11) | (10) |

SOP = $\bar{w}\bar{x} + yz$

# BINARY NUMBERS

1) Unsigned ] representation
2) Signed

## UNSIGNED

5 - 101
4 - 100

$$m = \sum_{i=0}^{n-1} x_i \times 2^i$$

$n$ = no. of digits
$x_i$ = digit

## SIGNED

MSB indicates sign

MSB 1 → -ve
MSB 0 → +ve

### Represent Positive No.s

+5 → 0101
+6 → 0110

### Represent Negative No.s
(three ways)

1. Simple Sign Magnitude

+5 → 0101
-5 → 1101

## 2. 1's complement

+5 → 0101

1's complement → 1010

invert all 0's and 1's

−5 → 1010

problem: +0 and −0 are different

+0 → 0000
−0 → 1111

## 3. 2's Complement

- we do not use sign magnitude or 1's complement forms
- 1's comp + 1

+5 → 0101
1's comp → 1010
−5 → 1011 → 2's complement

- advantage: 0 is universal

+0 → 0000
1's comp → 1111
−0 → 0000 (overflow)

- 2's complement of 2's complement is the original no.
- used in comps

- Eg: $+5 \longrightarrow 0101$

$\downarrow$ 2's complement

$-5 \longrightarrow 1011$

$\downarrow$ 2's complement

$+5 \longrightarrow 0101$

Intuition :

$-5 \longrightarrow 1011$

$-4 \longrightarrow 1100$ +1

$\therefore$ can add 2's comp to +ve no. for subtraction

- for an n-bit signed no, $2^{n-1}$ no.s for 0 & +ve, $2^{n-1}$ for -ve

| | | | |
|---|---|---|---|
| 0000 | 0 | 1000 | -8 |
| 0001 | 1 | 1001 | -7 |
| 0010 | 2 | 1010 | -6 |
| 0011 | 3 | 1011 | -5 |
| 0100 | 4 | 1100 | -4 |
| 0101 | 5 | 1101 | -3 |
| 0110 | 6 | 1110 | -2 |
| 0111 | 7 | 1111 | -1 |

# adding and subtracting binary numbers

## ADDITION

- binary addition

  5+3

$$\begin{array}{r} {\scriptstyle 1\,1\,1\ \ } \\ 0101 \\ +\ 0011 \\ \hline 1000 \end{array} = 8$$

## SUBTRACTION

- to do a-b do a + (-b) or a + 2's complement of b
- note: signed vs unsigned addition
- 3-5 = -5 + 3

$$\begin{array}{r} {\scriptstyle 1\ \ \ } \\ 1011 \\ +\ 0011 \\ \hline 1110 \end{array} \longrightarrow \text{find 2's comp}$$

0001 + 1 = 0010

∴ res = -2

## Example 12

- Add 0111 (7) and 1011 (11 or -5) — deal with overflow
  - ↪ signed
  - ↪ unsigned

unsigned

```
 1 11
  0111
+ 1011
 10010 = 2
```

overflow 18

signed

```
 1 11
  0111
+ 1011
 10010 = 2
```

no overflow?

There are 2 4-bit no.s whose 2's complement does not reverse the sign

1. 0000 → 0

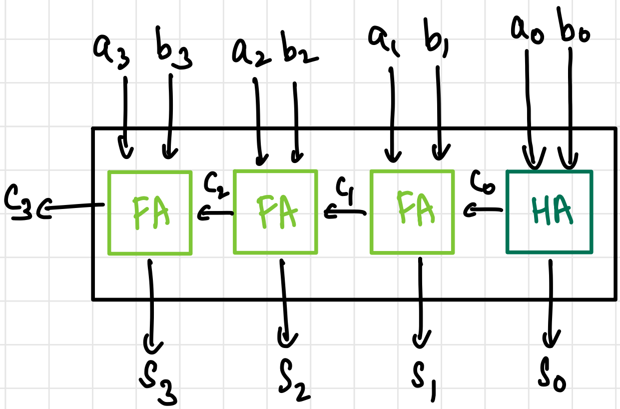
   2's comp: 1111 + 1 = 10000 (overflow)

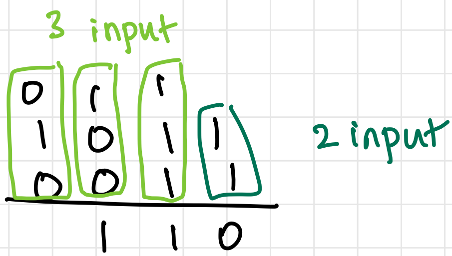
2. 1000 → -8

   2's comp: 0111 + 1 → 1000 → -8

# LOGIC CIRCUITS FOR ADDITION

## structure of logic circuits




- LSB: 2-input adder — half adder (HA)
- all other bits: 3-input adder — full adder (FA)

# HALF ADDER

## Truth Table

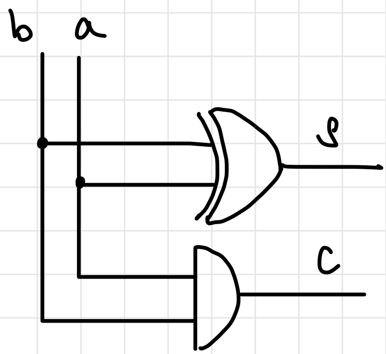
| inputs | | outputs | |
|---|---|---|---|
| a | b | s | c |
| 0 | 0 | 0 | 0 |
| 0 | 1 | 1 | 0 |
| 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 1 |

- SOP formulas (minterms)

$s = \bar{a}b + a\bar{b}$ (XOR function)

$s = a \oplus b$

$c = ab$ (AND function)



# FULL ADDER

| a | b | c | s | $C_{out}$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 | 0 |
| 0 | 1 | 0 | 1 | 0 |
| 0 | 1 | 1 | 0 | 1 |
| 1 | 0 | 0 | 1 | 0 |
| 1 | 0 | 1 | 0 | 1 |
| 1 | 1 | 0 | 0 | 1 |
| 1 | 1 | 1 | 1 | 1 |

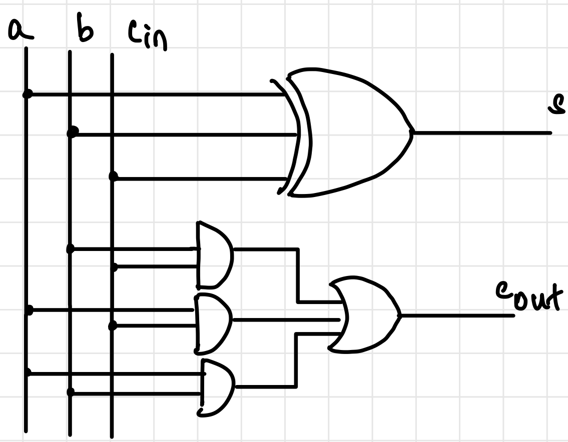
- SOP formulas

$$s = \bar{a}\bar{b}c + \bar{a}b\bar{c} + a\bar{b}\bar{c} + abc$$

$$s = a \oplus b \oplus c$$

$$c_{out} = \bar{a}bc + a\bar{b}c + ab\bar{c} + abc$$
$$= bc(a+\bar{a}) + ac(b+\bar{b}) + ab(c+\bar{c})$$

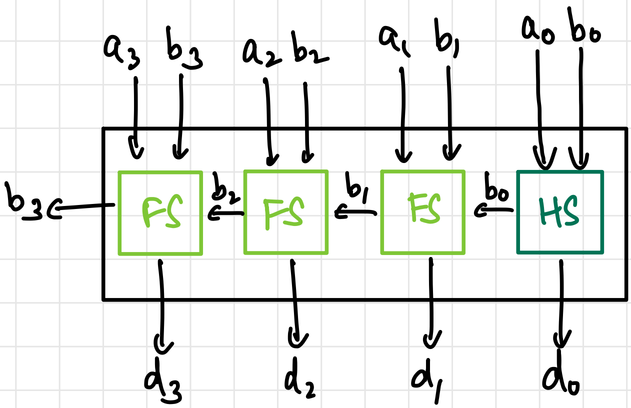
$$C_{out} = ab + bc + ac$$



## Example 13 — HW

Perform binary subtraction (not addition) and draw logic circuits

$$\begin{array}{rrrrr} & 0 & 1 & & \\ & \cancel{1} & \cancel{1}0 & 0 & 1 \\ - & 0 & 0 & 1 & 1 \\ \hline & 0 & 1 & 1 & 0 \end{array} \qquad \begin{array}{r} 9 \\ -3 \\ \hline 6 \end{array} \qquad \begin{array}{rrrrr} & 1 & 1 & & \\ & 1 & 0 & 0 & 1 \\ - & 0 & 0 & 1 & 1 \\ \hline & 0 & 1 & 1 & 0 \end{array}$$




$6 - 3 = 3$ $\qquad$ $5 - 2 = 3$

$$\begin{array}{rrrrr} & & 1 & 1 & \\ & 0 & 1 & 1 & 0 \\ - & 0 & 0 & 1 & 1 \\ \hline & 0 & 0 & 1 & 1 \end{array} \qquad \begin{array}{rrrrr} & & 1 & & \\ & 0 & 1 & 0 & 1 \\ - & 0 & 0 & 1 & 0 \\ \hline & 0 & 0 & 1 & 1 \end{array}$$

$a - b_{in} - b$

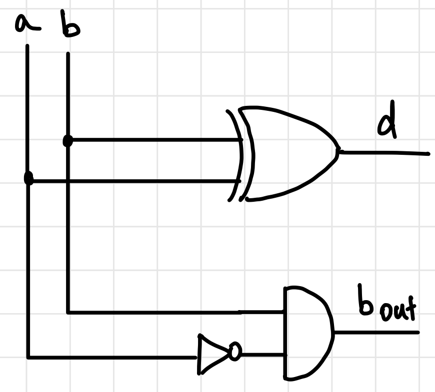
# HALF SUBTRACTOR

| a | b | d | $b_{out}$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 0 |

## SOP Formulas

$$d = \bar{a}b + a\bar{b} = a \oplus b$$

$$b_{out} = \bar{a}b$$



# FULL SUBTRACTOR

$a - b_{in} - b$

if $a < b + b_{in}$

2 is added to current digit

$a - b - b_{in}$
$= -1 \ (+2)$
$= 1$

| $a$ | $b$ | $c = b_{in}$ | $d$ | $b_{out}$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 | 1 |
| 0 | 1 | 0 | 1 | 1 |
| 0 | 1 | 1 | 0 | 1 |
| 1 | 0 | 0 | 1 | 0 |
| 1 | 0 | 1 | 0 | 0 |
| 1 | 1 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 | 1 |

$a - b - b_{in}$
$= -1 \ (+2)$
$= 1$

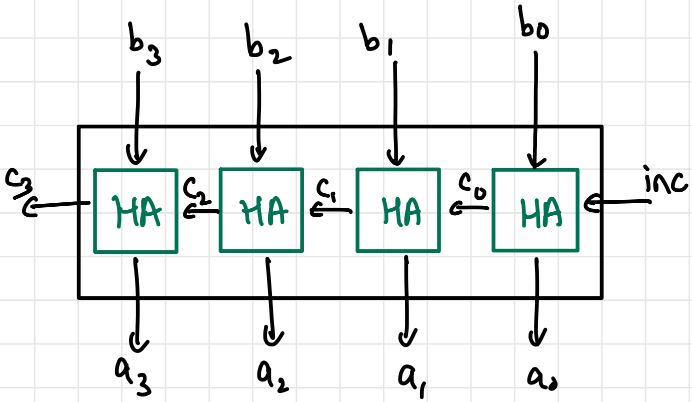
$$d = a \oplus b \oplus c$$

$$b_{out} = \bar{a}\bar{b}c + \bar{a}b\bar{c} + \bar{a}bc + abc$$
$$= bc(a+\bar{a}) + \bar{a}b(c+\bar{c}) + \bar{a}c(b+\bar{b})$$

$$b_{out} = bc + \bar{a}b + \bar{a}c$$

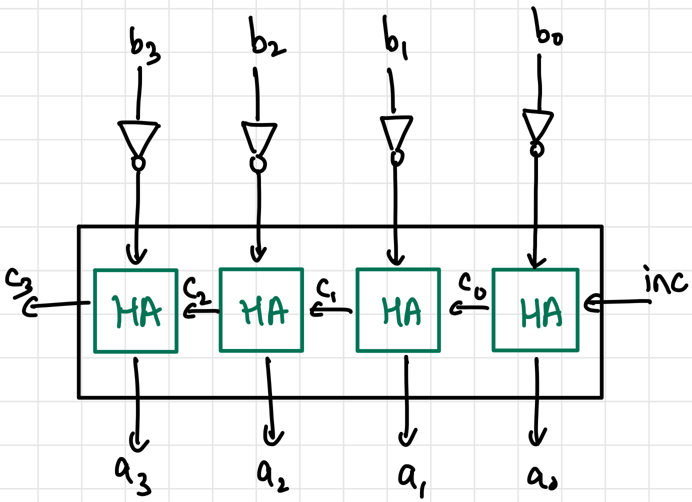
# INCREMENT LOGIC CIRCUIT

- b + 1

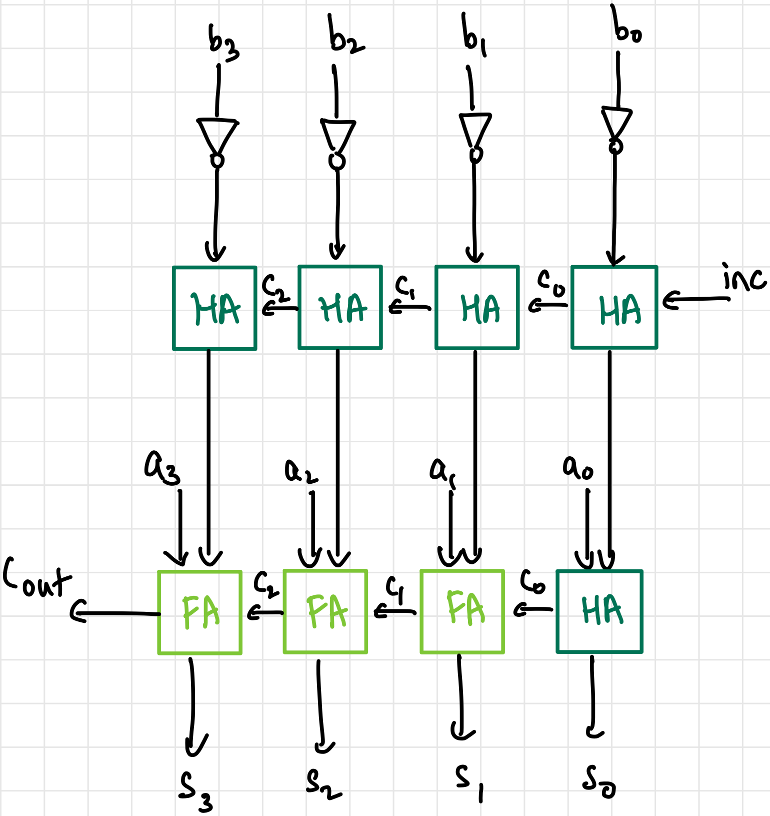


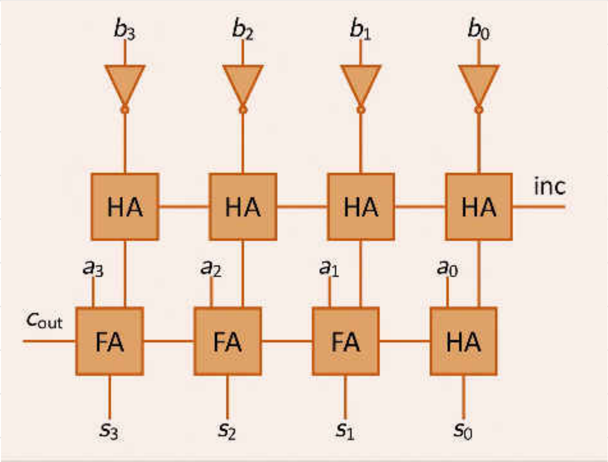

# TWO'S COMPLEMENT

- b inverted + 1



# TWO'S COMPLEMENT SUBTRACTOR

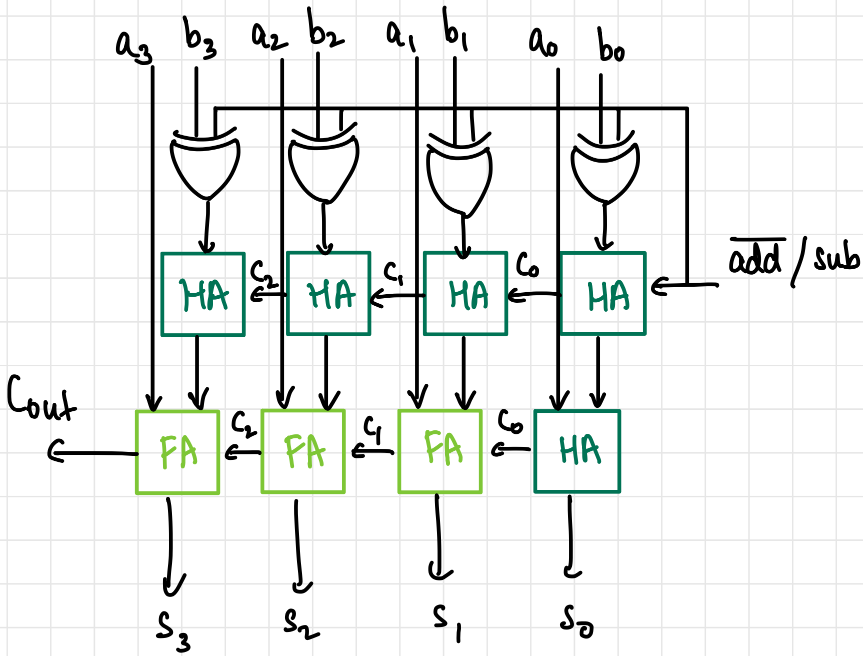


# TWO'S COMPLEMENT ADDER / SUBTRACTOR

## XOR as Controlled Inverter

| inv | a | y |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 0 |

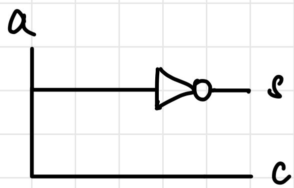


# INCREMENTOR WITHOUT INC INPUT

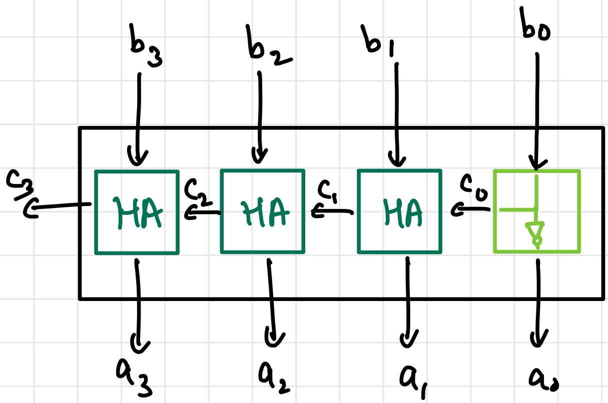
first incrementor

| a | s | c |
|---|---|---|
| 0 | 1 | 0 |
| 1 | 0 | 1 |

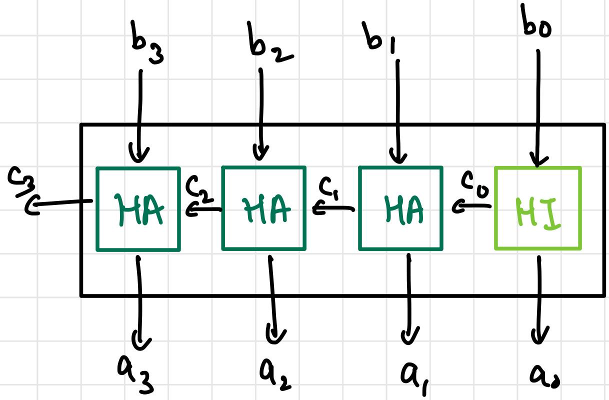
$s = \bar{a}$
$c = a$




half incrementor
HA




OR



# DECREMENTOR

- $b - 1 = b + (-1)$

## TWO'S COMPLEMENT DECREMENTOR

- $-1 = (1111)$

$$\begin{array}{rrrrrl} & & 1 & 1 & & \\ & 0 & 1 & 1 & 0 & 6 \\ + & 1 & 1 & 1 & 1 & +(-1) \\ \hline 1 & 0 & 1 & 0 & 1 & 5 \end{array}$$

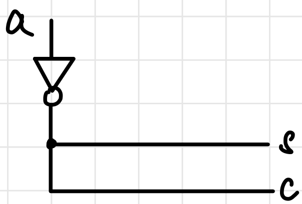
no overflow

$$\begin{array}{rrrrrl} & & & 1 & & \\ & 0 & 1 & 1 & 0 & 6 \\ - & 0 & 0 & 0 & 1 & -(1) \\ \hline & 0 & 1 & 0 & 1 & 5 \end{array}$$

## Half 2's Complement Decrementor

| a | s | c |
|---|---|---|
| 0 | 1 | 1 |
| 1 | 0 | 0 |

$s = \bar{a}$
$c = \bar{a}$



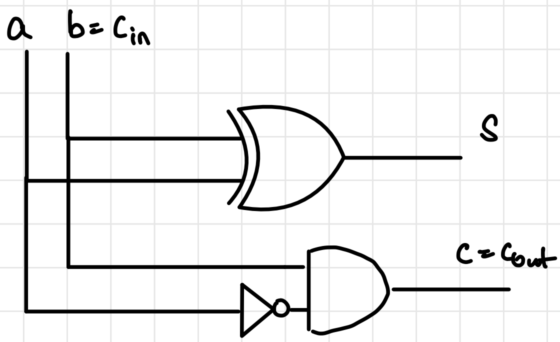
# Full 2's Complement Decrementor

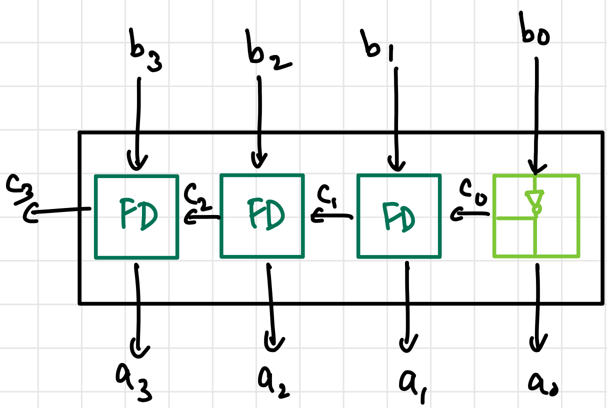
| a | Cin=b | S | Cout |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 0 |

$S = \bar{a}b + a\bar{b}$ (XOR)

$c = \bar{a}b$

Full decrementor FD

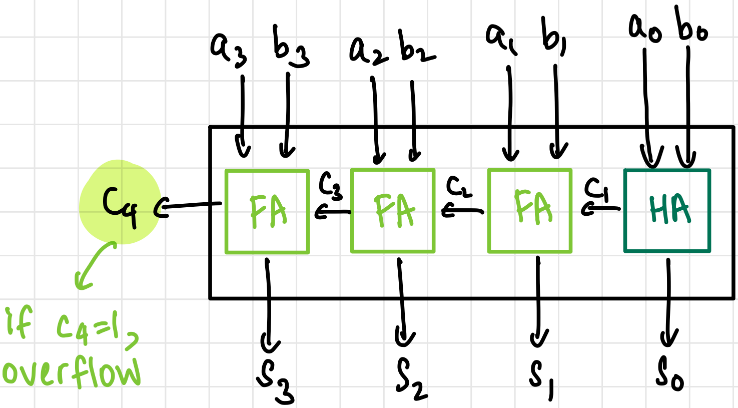


# Overflow

- Logic circuits have fixed bit widths
- Eg: 64 bit processor
- if result of an operation does not fit within the bit width

## SIMPLE ADDITION




## TWO'S COMPLEMENT ADDITION

- three cases

## Case I – no overflow

one +ve, one −ve
a +ve, b −ve

**case i**

$|a| > |b|$

result : +ve

**case ii**

$|b| > |a|$

result : −ve

**case iii**

$|a| = |b|$

result : 0

## Case II – overflow occurs

both +ve

↳ MSB is 1 ($C_{msb}$ is 1) ← should not happen in 2's comp +ve

↳ $C_{msb}$ is 1 and $C_{msb+1}$ is 0

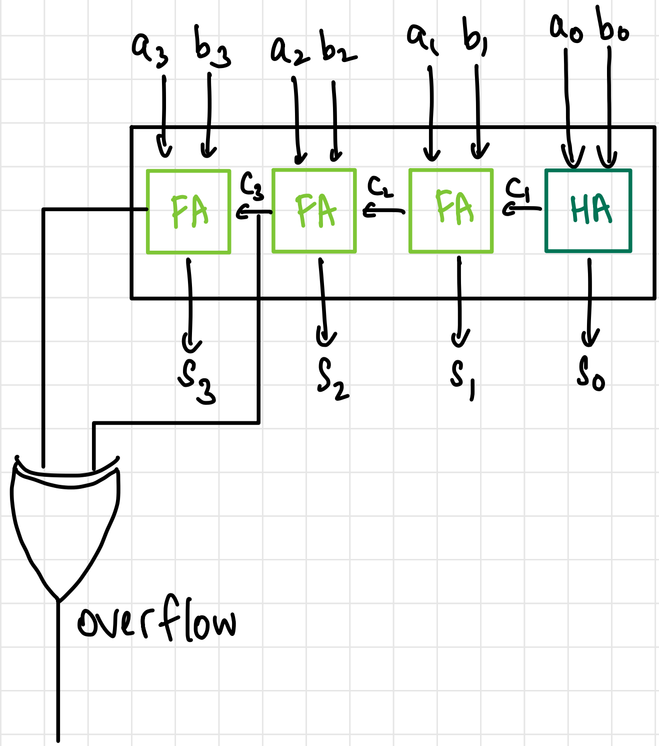
## Case III – overflow occurs

both −ve

↳ msb is 0 ($C_{msb}$ is 0)

↳ $C_{msb}$ is 0 and $C_{msb+1}$ is 1

$$\text{overflow} = C_{msb} \oplus C_{msb+1}$$


a3 b3 a2 b2 a1 b1 a0 b0
FA
c3
FA
c2
FA
c1
HA
S3 S2 S1 S0
overflow

## Question

Design combinational circuit for 3 i/p & 1 o/p.
The o/p is 1 when the binary value of the input is less than 3. The o/p is 0 otherwise

| a | b | c | y |
|---|---|---|---|
| 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 0 |
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 0 |

$$y = \bar{a}\bar{b}\bar{c} + \bar{a}\bar{b}c + \bar{a}b\bar{c}$$

$$y = \bar{a}\bar{b} + \bar{a}b\bar{c} + \bar{a}\bar{b}\bar{c}$$
$$= \bar{a}\bar{b} + \bar{a}\bar{c}$$

$$y = \bar{a}\bar{b} + \bar{a}\bar{c}$$

## Question

Design combinational circuit for 3 i/p & 1 o/p.
The o/p is 1 when the binary value of the input is even. The o/p is 0 otherwise

| a | b | c | y |
|---|---|---|---|
| 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 0 |
| 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 0 |
| 1 | 0 | 0 | 1 |
| 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 1 |
| 1 | 1 | 1 | 0 |

| a \ bc | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 0 | 1 (0) | 0 (1) | 0 (3) | 1 (2) |
| 1 | 1 (4) | 0 (5) | 0 (7) | 1 (6) |

$$y = \bar{c}$$

## Question

Design combinational circuit for 3 I/P x, y, z and 3 O/Ps a, b, c. When I/P is 0, 1, 2 or 3, O/P is one greater than I/P. When I/P is 4, 5, 7 or 7, O/P is 2 less than I/P.

| x | y | z | a | b | c |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 0 | 1 | 0 |
| 0 | 1 | 0 | 0 | 1 | 1 |
| 0 | 1 | 1 | 1 | 0 | 0 |
| 1 | 0 | 0 | 0 | 1 | 0 |
| 1 | 0 | 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 | 0 | 1 |

**a**

| $x \backslash yz$ | $\bar{y}\bar{z}$ 00 | $\bar{y}z$ 01 | $yz$ 11 | $y\bar{z}$ 10 |
|---|---|---|---|---|
| $\bar{x}$ 0 | 0 | 0 | 1 | 0 |
| $x$ 1 | 0 | 0 | 1 | 1 |

$$a = yz + xy$$

**b**

| $x \backslash yz$ | $\bar{y}\bar{z}$ 00 | $\bar{y}z$ 01 | $yz$ 11 | $y\bar{z}$ 10 |
|---|---|---|---|---|
| $\bar{x}$ 0 | 0 | 1 | 0 | 1 |
| $x$ 1 | 1 | 1 | 0 | 0 |

$$b = \bar{x}y\bar{z} + \bar{y}z + x\bar{y}$$

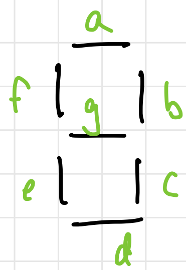
**b**

| $x \backslash yz$ | $\bar{y}\bar{z}$ 00 | $\bar{y}z$ 01 | $yz$ 11 | $y\bar{z}$ 10 |
|---|---|---|---|---|
| $\bar{x}$ 0 | 1 | 0 | 0 | 1 |
| $x$ 1 | 0 | 1 | 1 | 0 |

$$c = xz + \bar{x}\bar{z}$$

## Question

Logic circuit for 7-segment display




| a | b | c | d | a | b | c | d | e | f | g | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 |
| 0 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 2 |
| 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 3 |
| 0 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 4 |
| 0 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 5 |
| 0 | 1 | 1 | 0 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 6 |
| 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 7 |
| 1 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 8 |
| 1 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 9 |